PROMYS Europe 2026

Application Problem Set

https://www.promys-europe.org

Please attempt each of the following 8 problems. Though they can all be solved with no more than a standard high school mathematics background, most of the problems require considerably more ingenuity than is usually expected in high school. You should keep in mind that we do not expect you to find complete solutions to all of them. Rather, we are looking to see how you approach challenging problems.

We ask that you tackle these problems by yourself. Please see below for further information on citing any sources you use in your explorations.

Here are a few suggestions:

- Think carefully about the meaning of each problem.
- Examine special cases, either through numerical examples or by drawing pictures.
- Be bold in making conjectures.
- Test your conjectures through further experimentation, and try to devise mathematical proofs to support the surviving ones.
- Can you solve special cases of a problem, or state and solve simpler but related problems?

If you think you know the answer to a question, but cannot prove that your answer is correct, tell us what kind of evidence you have found to support your belief. If you use books, articles, or websites in your explorations, be sure to cite your sources.

The use of AI tools is not permitted. Any application suspected of using AI-generated content may be rejected. While general online resources are not prohibited if appropriately cited, be careful if you search online for help. We are interested in your ideas, not in solutions that you have found elsewhere. If you search online for a problem and find a solution (or most of a solution), it will be much harder for you to demonstrate your insight to us.

You may find that most of the problems require some patience. Do not rush through them. It is not unreasonable to spend a month or more thinking about the problems. It might be good strategy to devote most of your time to a small selection of problems which you find especially interesting. Be sure to tell us about progress you have made on problems not yet completely solved. **For each problem you solve, please justify your answer clearly and tell us how you arrived at your solution: this includes your experimentation and any thinking that led you to an argument.**

There are various tools available for typesetting mathematics on a computer. You are welcome to use one of these if you choose, or you are welcome to write your solutions by hand, or you might want to do a bit of both. We are interested in your mathematical ideas, not in your typesetting. What is important is that we can read all of your submitted work, and that you can include all of your ideas (including the ones that didn't fully work).

1. Calculate each of the following:

$$\begin{aligned} 1^3 + 5^3 + 3^3 &= ?? \\ 16^3 + 50^3 + 33^3 &= ?? \\ 166^3 + 500^3 + 333^3 &= ?? \\ 1666^3 + 5000^3 + 3333^3 &= ?? \end{aligned}$$

What do you see? Can you state and prove a generalization of your observations?

2. The sequence (x_n) of positive real numbers satisfies the relationship $x_{n-1}x_nx_{n+1} = 1$ for all $n \geq 2$. If $x_1 = 1$ and $x_2 = 2$, what are the values of the next few terms? What can you say about the sequence? What happens for other starting values?

The sequence (y_n) satisfies the relationship $y_{n-1}y_{n+1} + y_n = 1$ for all $n \geq 2$. If $y_1 = 1$ and $y_2 = 2$, what are the values of the next few terms? What can you say about the sequence? What happens for other starting values?

3. To get the *rerun* of a positive integer, we write it twice in a row without a space. For example the rerun of 2026 is 20262026. Is there a positive integer whose rerun is a perfect square? If so, how many such positive integers can you find? Are there infinitely many such positive integers? If not, then why not?

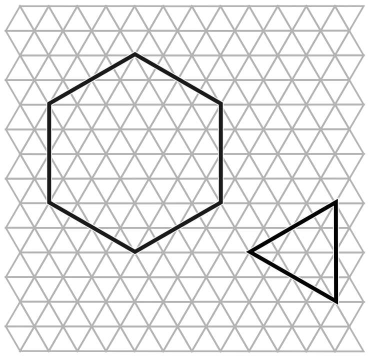
4. A few hikers set off along a trail that stretches infinitely ahead. The path is so narrow that overtaking is impossible. Each hiker walks at a distinct constant speed. Whenever a faster hiker catches up with a slower one ahead, they form a hiking group and continue together at the slower walker's pace. Initially, three hikers are equally spaced along the trail. After enough time has passed, some hikers may have merged into groups. Suppose the same three hikers began in a different order along the trail, but each kept their original starting speed. On average (averaging over all possible initial orderings), how many groups will remain after a long time? How does the answer change if there are 4 hikers? 5 hikers? In general, what happens for n hikers?

5. A particular two-player game starts with a pile of emeralds and a pile of pearls. On your turn, you can take any number of emeralds, or any number of pearls, or an equal number of each. You must take at least one gem on each of your turns. Whoever takes the last gem wins the game. For example, in a game that starts with 5 emeralds and 10 pearls, a game could look like: you take 2 emeralds, then your opponent takes 7 pearls, then you take 3 emeralds and 3 pearls to win the game. You get to choose the starting number of emeralds and pearls, and whether you go first or second. Find all starting configurations (including who goes first) with 8 gems where you are guaranteed to win. If you have to let your opponent go first, what are the starting configurations of gems where you are guaranteed to win? If you can't find all such configurations, describe the ones you do find and any patterns you see.

6. Suppose b and c are positive integers with $2025 < b < c$ and suppose $2025, b,$ and c have no common factors. Let S be the set of all integers of the form $2025x + by + cz$ where x, y, z are non-negative integers and let T be the set of all positive integers *not* in S. It turns out that T is a finite set.

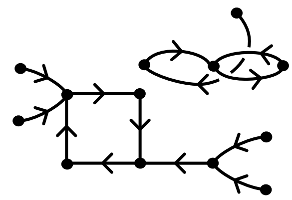
Let t_o and t_e denote, respectively, the numbers of odd and even integers in T. How big can the difference $|t_o - t_e|$ be? Prove your result and give an explicit combination of b and c that achieves this maximum.

7. The plane is tiled with equilateral triangles of side length 1, forming a triangular grid like the one shown in the picture.



The figure shows that it is possible to form a hexagon and an equilateral triangle whose vertices lie on points of the triangular grid. Is it possible to form a square whose vertices also lie on points of the grid? If not, is it possible to draw a square with side length at least 1 such that each of its vertices is at most a distance of $\frac{1}{10}$ away from some grid point? What if we require the distance to be at most $\frac{1}{100}$?

8. Suppose k points are joined by k nonoverlapping arrows in such a way that each point is the source of exactly one arrow. We call such a figure a *k-doodle*. Here are four examples



(a) (b) (c) (d)

The first three examples are loops, while the fourth is a 13-doodle with two components. This 13-doodle contains two loops. Does every k-doodle contain a loop? By moving one arrow, we can redraw this 13-doodle in the plane. Can every k-doodle be redrawn in the plane (without crossings) by moving some of its arrows around?